

Dynamic Adverse Selection and Liquidity*

Ioanid Roşu[†]

February 1, 2019

Abstract

Does a larger fraction of informed trading generate more illiquidity, as measured by the bid-ask spread? We answer this question in the negative in the context of a dynamic dealer market where the fundamental value follows a random walk, provided we consider the long run (stationary) equilibrium. More informed traders tend to generate more adverse selection and hence larger spreads, but at the same time cause faster learning by the market makers and hence smaller spreads. These two effects offset each other in the long run.

KEYWORDS: Learning, adverse selection, dynamic model, stationary distribution.

*We thank Bruno Biais, Denis Gromb, Augustin Landier, Stefano Lovo, Talis Putnins, and Daniel Schmidt for their suggestions. We are also grateful to finance seminar participants at HEC Paris and the Bucharest Academy of Economic Studies, as well as conference participants at the AFFI/Eurofidai conference in Paris for valuable comments.

[†]HEC Paris, Email: rosu@hec.fr.

1 Introduction

A traditional view of market liquidity, going at least as far back as Bagehot (1971), posits that one of the causes of illiquidity is adverse selection: “*The essence of market making, viewed as a business, is that in order for the market maker to survive and prosper, his gains from liquidity-motivated transactors must exceed his losses to information motivated transactors. [...] The spread he sets between his bid and asked price affects both: the larger the spread, the less money he loses to information-motivated, transactors and the more he makes from liquidity-motivated transactors.*”

This intuition has later been made precise by models such as Glosten and Milgrom (1985, henceforth GM85), in which a competitive risk-neutral dealer sequentially sets bid and ask prices in a risky asset, and makes zero expected profits in each trading round. Traders are selected at random from a population that contains a fraction ρ of informed traders, and must trade at most one unit of the asset. The asset liquidates at a value v that is constant and is either zero or one. In equilibrium, the bid-ask spread is wider when the informed share ρ is higher: there is more adverse selection, hence the dealer must set a larger bid-ask spread to break even.

This intuition, however, must be modified once we consider the dynamics of the bid-ask spread. A larger informed share also means that orders carry more information, which over time reduces the uncertainty about v and thus puts downward pressure on the bid-ask spread. We call this last effect *dynamic efficiency*. This effect is already present in GM85, who observe that a larger informed share causes initially a larger bid-ask spread, but also causes the bid-ask spread to decrease faster to its eventual value, which is zero (when v is fully learned).

A natural question is then: to what extent does dynamic efficiency reduce the traditional adverse selection? To answer this question, we extend the framework of GM85 to allow v to move over time according to a random walk v_t .¹ To obtain closed-form results, we assume that the increments of v_t are normally distributed with volatility σ_v , called the *fundamental*

¹In GM85 both types of traders are willing to trade in each period (the informed because v is always outside the bid-ask spread, the uninformed for exogenous reasons), hence a trader is chosen at random among the informed and uninformed. When v_t is moving, there are times when the informed traders are not willing to trade (v_t is within the spread), hence a trader is chosen at random among the uninformed.

volatility. We chose a moving value for two reasons. First, this is a realistic assumption in modern financial markets, where relevant information arrives essentially at a continuous rate. Second, we want to study the long-term evolution of the bid-ask spreads, and this long-term analysis is trivial when v_t is constant, as the dealer eventually fully learns v . Note that we are interested in the long run because (as we show later) in the short run the equilibrium is similar to GM85, but in the long run it converges to the *stationary* equilibrium, which has novel properties.

The first property of the stationary equilibrium is that the dealer's uncertainty about v_t is constant. More precisely, we define the *public density* at t to be the dealer's posterior density of v_t just before trading at t . We also define the *public mean* and *public volatility* to be, respectively, the mean and standard deviation of the public density.² Thus, in a stationary equilibrium, the public volatility (which is a measure of the dealer's uncertainty) is constant. The second property of the stationary equilibrium is that the informed share is inversely related to the public volatility. The intuition is simple: when the informed share is low, the order flow carries little information, and thus the public volatility is large.³

A surprising property of the stationary equilibrium is that the informed share has no effect on the bid-ask spread. To understand this result, consider a small informed share, say 1%. Suppose a buy order arrives, and the dealer estimates how much to update the public mean (in equilibrium this update is half of the bid-ask spread). There are two opposite effects. First, it is very unlikely that the buy order comes from an informed trader (with only 1% chance). This is the *adverse selection effect*: a low informed share makes the dealer less concerned about adverse selection, which leads to a smaller update of the public mean, and hence decreases the bid-ask spread. But, second, if the buy order does come from an informed trader, a large public volatility translates into the dealer knowing that, on average, the informed trader must have observed a value far above the public mean. This is the *dynamic efficiency effect*: a low informed share leads to a larger update of the public mean, and hence increases the bid-ask

²To simplify the analysis, we assume here (as in Section 4) that the dealer always considers the public density to be normal: after observing the order flow at t , the dealer computes correctly the first two moments of the public density at $t + 1$, but not the higher moments, and thus considers the new density to be normal as well. In Section 5, we show that the main results remain robust with exact learning.

³The existence itself of a stationary limit is not entirely obvious. Indeed, it might be possible for the public volatility to grow indefinitely, without any finite limit.

spread. At the other end, a large informed share means that the dealer learns well about the asset value (the public volatility is small), and therefore the bid-ask spread tends to be small.

It turns out that the two opposite effects exactly offset each other. As explained above, this translates into the fact that the magnitude of the updates in public mean caused by order flow is independent of the informed share. This result depends crucially on the equilibrium being stationary. To understand why, consider an equilibrium which is not necessarily stationary. If there was no order flow at t , then the dealer's uncertainty (the public volatility) would increase from t to $t + 1$ as the asset value diffuses. But the order flow at t contains information and hence reduces the uncertainty at $t + 1$. In a stationary equilibrium the uncertainty increase caused by diffusion must cancel the uncertainty decrease caused by order flow. Thus, as the value diffusion is independent of the informed share, the information content of the order flow must also be independent of the informed share. But this implies that the magnitude of public mean updates is independent of the informed share.

Our next result is that, for any initial public volatility, the equilibrium converges to the stationary equilibrium. In particular, consider a wide initial public volatility. Then, as the order flow starts providing information to the dealer, the public volatility starts decreasing toward its stationary value. The same is true for the bid-ask spread, which in a non-stationary equilibrium is always proportional to the public volatility. This phenomenon is similar to the GM85 equilibrium, except that there the stationary public volatility and bid-ask spread are both zero. This illustrates the statement made above, that the non-stationary equilibrium (the "short run") resembles GM85, while the stationary equilibrium (the "long run") is different and produces novel insights.

Studying the equilibrium behavior after various types of shocks provides a few testable implications. First, consider a positive shock to the informed share (e.g., the stock is now studied by more hedge funds). Then, the adverse selection effect suddenly becomes stronger, and as a result the bid-ask spread temporarily increases. In the long run, though, the bid-ask spread reverts to its stationary value, which does not change. At the same time, the public volatility gradually decreases to its new level, which is lower due to the increase in the informed share. Second, consider a negative shock to the current public volatility (e.g., public

news about the current asset value). Then, the bid-ask spread follows the public volatility and drops immediately, after which it increases gradually to its old stationary level. Third, consider a positive shock to the fundamental volatility (e.g., all future uncertainty about the asset increases). Then, the bid-ask spread follows the public volatility and increases gradually to its new stationary level.

Based on our results, the picture on dynamic adverse selection that emerges is that liquidity is more strongly affected not by the informed share (the intensive margin), but by the fundamental volatility (the extensive margin). By contrast, price discovery (measured by the public volatility) is strongly affected by both informed share and fundamental volatility. This suggests that the presence of privately informed traders can be more precisely identified by proxies of the current level of uncertainty, rather than by illiquidity measures such as the bid-ask spread (which is used by Collin-Dufresne and Fos, 2015).

A surprising outcome of our theory is that a lower level of uncertainty (lower public volatility) can occur if either the informed share becomes larger (more privately informed traders arrive), or more precise public news arrives.⁴ We can disentangle the two scenarios, however, by examining the effect on the bid-ask spread: more precise public news should reduce it, while a larger informed share should have no effect.

Our paper contributes to the literature of dynamic models of adverse selection.⁵ To our knowledge, this paper is the first to study the effect of stationarity in dealer models of the Glosten and Milgrom (1985) type.⁶ By contrast, several stationary models of the Kyle (1985) type are analyzed for instance by Chau and Vayanos (2008) and Caldentey and Stacchetti (2010). The focus of these models, however, is not liquidity but price discovery: it turns out that in the limit the market in these models becomes strong-form efficient, as the insider trades very aggressively.

The paper speaks to the literature on the identification of informed trading and in particular on the identification of insider trading. Collin-Dufresne and Fos (2015, 2016) show

⁴In Section 6.2 we solve a simple extension of our model with public news, and show that more precise public news translates into a lower public volatility.

⁵See for instance the survey of Foucault, Pagano, and Röell (2013) and the references therein.

⁶Glosten and Putnins (2016) study the welfare effect of the informed share in the Glosten and Milgrom (1985) model, but they do not consider the effect of stationarity.

both empirically and theoretically that times when insiders trade coincide with times when liquidity is actually stronger (and in particular bid-ask spreads decline). They attribute this finding to the action of discretionary insiders who trade when they expect a larger presence of liquidity (noise) traders. During those times the usual positive effect of noise traders on liquidity dominates, and thus bid-ask spreads decline despite there being more informed trading. By contrast, our effect works even when the noise trader activity is constant over time, as long as there is enough time for the equilibrium to become stationary.

The paper is organized as follows. Section 2 describes the model (in which the value follows a random walk). Section 3 shows how to compute the equilibrium when the dealer is fully Bayesian. Section 4 studies in detail the model in which the dealer is approximately Bayesian, and describes the stationary and non-stationary equilibria. Section 5 verifies how well the approximate equilibrium approaches the exact equilibrium. Section 7 concludes. All proofs are in the Appendix. The Internet Appendix contains a discussion of general dealer models, and an application to a model in which the fundamental value switches randomly between zero and one.

2 Environment

The model is similar to GM85, except that the fundamental value moves according to a random walk:

$$v_{t+1} = v_t + \varepsilon_{t+1}, \quad \text{with } \varepsilon_t \stackrel{IID}{\sim} \mathcal{N}(\cdot, 0, \sigma_v). \quad (1)$$

There is a single risky asset, and time is discrete and infinite. Trading in the risky asset is done on an exchange, where before each time $t = 0, 1, 2, \dots$ a dealer posts two quotes: the *ask price* (or simply *ask*) A_t , and the *bid price* (or simply *bid*) B_t . Thus, a buy order at t executes at A_t , while a sell order at t executes at B_t . The dealer (referred to in the paper as “she”) is risk neutral and competitive, and therefore makes zero expected profits from each trade.

The buy or sell orders are submitted by a trading population with a fraction $\rho \in (0, 1)$ of informed traders and a fraction $1 - \rho$ of uninformed traders. At each $t = 0, 1, \dots$ a trader is selected at random from the population willing to trade, and can trade at most one unit of

the asset. An uninformed trader at t is always willing to trade, and is equally likely to buy or to sell.⁷ An informed trader at t who observes the value v_t either (i) submits a buy order if $v_t > A_t$, (ii) submits a sell order if $v_t < B_t$, (iii) is not willing to trade if $v_t \in [B_t, A_t]$. If case (iii) occurs, an uninformed trader is selected, as no informed trader is willing to trade.

The dealer's uncertainty about the fundamental value is summarized by the *public density*, which is the density of v_t just before trading at t , conditional on all the order flow available at t , that is, the sequence of orders submitted at times $0, 1, \dots, t-1$. Denote by ϕ_t the public density, by μ_t its mean (called the *public mean*) and by σ_t its standard deviation (called the *public volatility*). The initial density ϕ_0 is assumed to be rapidly decaying at infinity.⁸ In the rest of the paper, by "density" we typically include the requirement that the density be rapidly decaying. To avoid cumbersome language, we make this requirement explicit only when we state the formal results.

3 Equilibrium

We prove the existence of an equilibrium of the model in two steps. First, for each $t = 0, 1, 2, \dots$ we start with an public density ϕ_t , an ask A_t , a bid $B_t < A_t$, and compute the public density ϕ_{t+1} after a buy or sell order. Second, for any public density ϕ_t we show that there exists an *ask-bid pair* (A_t, B_t) , meaning that the ask A_t and the bid B_t satisfy the dealer's pricing conditions which require that her expected profit from trading at t is zero. The ask-bid pair (A_t, B_t) is not necessarily unique, and we choose the pair with the ask closest to the public mean.

⁷A standard way to endogenize this assumption is to introduce relative private valuations for the uninformed traders. For instance, if a trader expects the value to be μ_t and has a relative private valuation larger than $A_t - \mu_t$ (which in equilibrium is half the bid-ask spread), the trader is always willing to buy at A_t .

⁸A function f is rapidly decaying (at infinity) if it is smooth and satisfies $\lim_{v \rightarrow \pm\infty} |v|^M f_0^{(N)}(v) = 0$, where $f^{(N)}$ is the N -th derivative of f . The space \mathcal{S} of rapidly decaying functions is called the Schwartz space. Any normal density belongs to \mathcal{S} , and the convolution of two densities in \mathcal{S} also belongs to \mathcal{S} .

3.1 Evolution of the Public Density

Let ϕ_t be the public density of v_t before trading at t , and let $A_t > B_t$ be, respectively, the ask and bid at t (not necessarily satisfying the dealer's pricing conditions). Suppose a buy or sell order $\mathcal{O}_t \in \{\text{B}, \text{S}\}$ arrives at t . Let $\mathbf{1}_P$ be the indicator function, which is one if P is true and zero if P is false. Conditional on $v_t = v$, the probability of observing the a buy order at t is

$$g_t(\text{B}, v) = \rho \mathbf{1}_{v > A_t} + \frac{\rho}{2} \mathbf{1}_{v \in [B_t, A_t]} + \frac{1-\rho}{2}. \quad (2)$$

To see this, consider the following cases:

- If $v \in [B_t, A_t]$, the informed traders are not willing to trade, and an uninformed trader submits a buy order with probability $\frac{1}{2}$. Then, $g_t(\text{B}, v) = \rho \times 0 + \frac{\rho}{2} \times 1 + \frac{1-\rho}{2} = \frac{1}{2}$.
- If $v \notin [B_t, A_t]$, an informed trader (chosen with probability ρ) submits a buy order with probability $\mathbf{1}_{v > A_t}$, while an uninformed trader (chosen with probability $1 - \rho$) submits a buy order with probability $\frac{1}{2}$. Then, $g_t(\text{B}, v) = \rho \mathbf{1}_{v > A_t} + \frac{\rho}{2} \times 0 + \frac{1-\rho}{2}$

Similarly, the probability of observing a sell order at t is

$$g_t(\text{S}, v) = \rho \mathbf{1}_{v < B_t} + \frac{\rho}{2} \mathbf{1}_{v \in [B_t, A_t]} + \frac{1-\rho}{2}. \quad (3)$$

The next result describes the evolution of the public density.

Proposition 1. *Consider a rapidly decaying public density ϕ_t , and an ask-bid pair with $A_t > B_t$. After observing an order $\mathcal{O}_t \in \{\text{B}, \text{S}\}$, the density of v_t is $\psi_t(v|\mathcal{O}_t)$, where*

$$\begin{aligned} \psi_t(v|\text{B}) &= \frac{(\rho \mathbf{1}_{v > A_t} + \frac{\rho}{2} \mathbf{1}_{v \in [B_t, A_t]} + \frac{1-\rho}{2}) \cdot \phi_t(v)}{\frac{\rho}{2}(1 - \Phi_t(A_t)) + \frac{\rho}{2}(1 - \Phi_t(B_t)) + \frac{1-\rho}{2}}, \\ \psi_t(v|\text{S}) &= \frac{(\rho \mathbf{1}_{v < B_t} + \frac{\rho}{2} \mathbf{1}_{v \in [B_t, A_t]} + \frac{1-\rho}{2}) \cdot \phi_t(v)}{\frac{\rho}{2}\Phi_t(A_t) + \frac{\rho}{2}\Phi_t(B_t) + \frac{1-\rho}{2}}, \end{aligned} \quad (4)$$

where Φ_t is the cumulative density function corresponding to ϕ_t . The public density at $t + 1$

is rapidly decaying, and satisfies

$$\phi_{t+1}(w|\mathcal{O}_t) = \int_{-\infty}^{+\infty} \psi_t(v|\mathcal{O}_t)\mathcal{N}(w-v, 0, \sigma_v)dv = \left(\psi_t(\cdot|\mathcal{O}_t) * \mathcal{N}(\cdot, 0, \sigma_v)\right)(w), \quad (5)$$

where “*” denotes the convolution of two densities.

Proposition 1 shows how the public density evolves once a particular order (buy or sell) is submitted at t . Note, however, that this result does not assume anything about the ask and bid other than $A_t > B_t$, so in principle these can be chosen arbitrarily. In equilibrium, however, these prices must satisfy the dealer’s pricing conditions, namely that the dealer’s expected profits at t must be zero.

In the next section (Section 3.2) we impose these conditions and we show how to determine the equilibrium ask and bid. Then, Proposition 1 allows us to describe the whole evolution of the public density, conditional on the initial density ϕ_0 and the sequence of orders $\mathcal{O}_0, \mathcal{O}_1, \dots$ that have been submitted.

3.2 Ask and Bid Prices

Let ϕ_t be the public density of v_t before trading at t . We define an *ask-bid pair* (A_t, B_t) as a pair of ask and bid satisfying the pricing conditions of the dealer. As the dealer is risk neutral and competitive, the pricing conditions are: (i) the ask A_t is the expected value of v_t conditional on a buy order at t , and (ii) the bid B_t is the expected value of v_t conditional on a sell order at t . Using the previous notation, the dealer’s pricing conditions are that A_t is the mean of $\psi_t(v|B)$, the posterior density of v_t after observing a buy order at t ; and B_t is the mean of $\psi_t(v|S)$, the posterior density after observing a sell order at t . Thus, the dealer’s pricing conditions are equivalent to

$$A_t = \int_{-\infty}^{+\infty} v\psi_t(v|B)dv, \quad B_t = \int_{-\infty}^{+\infty} v\psi_t(v|S)dv. \quad (6)$$

For future use, we record the following straightforward result.

Corollary 1. *The pair (A_t, B_t) is an ask-bid pair if and only if the following equations are satisfied:*

$$A_t = \mu_{t+1,B}, \quad B_t = \mu_{t+1,S}, \quad \text{with} \quad \mu_{t+1,\mathcal{O}_t} = \int_{-\infty}^{+\infty} w\phi_{t+1}(w|\mathcal{O}_t)dw, \quad \mathcal{O}_t = \{B, S\}. \quad (7)$$

The next result shows that the existence of an ask-bid pair is equivalent to solving a 2×2 system of nonlinear equations. Suppose μ_t is the mean of ϕ_t . For $(A, B) \in (\mu_t, \infty) \times (-\infty, \mu_t)$, define the functions:

$$\begin{aligned} F(A, B) &= \frac{\Theta_t(A) + \Theta_t(B)}{A - \mu_t} - \frac{1 + \rho}{\rho} + \Phi_t(A) + \Phi_t(B), \\ G(A, B) &= \frac{\Theta_t(A) + \Theta_t(B)}{\mu_t - B} - \frac{1 - \rho}{\rho} - \Phi_t(A) - \Phi_t(B), \end{aligned} \quad (8)$$

where Φ_t is the cumulative density associated to ϕ_t , and Θ_t is defined by

$$\Theta_t(v) = \int_{-\infty}^v (\mu_t - w)\phi_t(w)dw. \quad (9)$$

The function Θ_t is strictly positive everywhere and approaches zero at infinity on both sides.⁹

Proposition 2. *Consider a rapidly decaying public density ϕ_t , with mean μ_t . Then, the existence of an ask-bid pair is equivalent to finding a solution $(A, B) \in (\mu_t, \infty) \times (-\infty, \mu_t)$ of the system of equations:*

$$F(A, B) = 0, \quad G(A, B) = 0. \quad (10)$$

A solution of (10) always exists. Among the set of ask-bid pairs (A, B) there is a unique one for which A is closest to μ_t .

The last statement in Proposition 2 shows that one can choose a unique ask-bid pair based on the criterion that the ask A be the closest to the public mean μ_t . Denote this pair by (A_t, B_t) . In the rest of the paper, we assume that this is indeed the ask-bid pair chosen by the dealer.¹⁰

⁹As ϕ_t is rapidly decaying, $\Theta_t(-\infty)$ is equal to zero. The definition of μ_t implies that $\Theta_t(+\infty) = \int_{-\infty}^{+\infty} (\mu_t - w)\phi_t(w)dw = \mu_t - \int_{-\infty}^{+\infty} w\phi_t(w)dw = 0$. Also, $\Theta_t'(v) = (\mu_t - v)\phi_t(v)$, hence $\Theta_t(v)$ is increasing below μ_t and decreasing above μ_t . As $\Theta_t(\pm\infty) = 0$, the function Θ_t is strictly positive everywhere.

¹⁰In principle, the equations in (10) might have multiple solutions, meaning that one could manufacture an

4 Equilibrium with Approximate Bayesian Inference

In this section, we assume that at each step the dealer approximates the public density with a normal density such that the first two moments are correctly computed. Specifically, suppose that the dealer regards v_t to be distributed as

$$\phi_t^a(v) = \mathcal{N}(v, \mu_t, \sigma_t). \quad (11)$$

After the dealer observes an order \mathcal{O}_t at t , denote by $\phi_{t+1}(w|\mathcal{O}_t)$ the exact density of v_{t+1} conditional on the past order flow including \mathcal{O}_t , and by μ_{t+1, \mathcal{O}_t} and $\sigma_{t+1, \mathcal{O}_t}$ its mean and standard deviation, respectively. Then, before trading at $t + 1$ the dealer regards v_{t+1} to be distributed as

$$\phi_{t+1}^a(w|\mathcal{O}_t) = \mathcal{N}(w, \mu_{t+1, \mathcal{O}_t}, \sigma_{t+1, \mathcal{O}_t}). \quad (12)$$

Thus, we assume that the dealer continues to make the approximation at each step:

$$\phi_t = \phi_t^a. \quad (13)$$

Section 5 discusses the accuracy of this approximation. For simplicity, we continue to refer to $\phi_t(v)$ as the *public density*, μ_t as the *public mean*, and σ_t as the *public volatility*.

4.1 Evolution of the Public Density

Proposition 3. *Suppose the public density at $t = 0, 1, 2, \dots$ is $\phi_t(v) = \mathcal{N}(v, \mu_t, \sigma_t)$. After observing $\mathcal{O}_t \in \{B, S\}$, the posterior mean and volatility at $t + 1$ satisfy*

$$\mu_{t+1, B} = \mu_t + \delta\sigma_t, \quad \mu_{t+1, S} = \mu_t - \delta\sigma_t, \quad \sigma_{t+1, B} = \sigma_{t+1, S} = \sqrt{(1 - \delta^2)\sigma_t^2 + \sigma_v^2}. \quad (14)$$

public density ϕ_t for which there is more than one corresponding ask-bid pair. Numerically, we have computed the sequence of public densities that starts with a normal density ϕ_0 and is associated by an arbitrary sequence of orders, but we have not yet been able to encounter a non-unique ask-bid pair. Nevertheless, we must account for the possibility that such non-uniqueness may in fact arise.

where δ is defined by:

$$\delta = g^{-1}(2\rho), \quad \text{with } g(x) = \frac{x}{\mathcal{N}(x, 0, 1)}. \quad (15)$$

There is a unique ask: $A_t = \mu_t + \delta\sigma_t$ and unique bid: $B_t = \mu_t - \delta\sigma_t$, and the bid-ask spread is

$$s_t = A_t - B_t = 2\delta\sigma_t. \quad (16)$$

We now investigate whether the public density reaches a steady state, in the sense that its shape converges to a particular density. As the mean μ_t evolves according to a random walk, we must demean the public density and focus on its standard deviation σ_t . The next result shows that the public volatility σ_t converges to a particular value, σ_* , regardless of the initial value σ_0 .

Proposition 4. *For any $t = 0, 1, 2, \dots$ the public volatility satisfies*

$$\sigma_t^2 = \sigma_*^2 + (\sigma_0^2 - \sigma_*^2)(1 - \delta^2)^t, \quad (17)$$

where

$$\sigma_* = \frac{\sigma_v}{\delta} = \frac{\sigma_v}{g^{-1}(2\rho)}. \quad (18)$$

For any initial value σ_0 and any sequence of orders, the public volatility σ_t monotonically converges to σ_* , and the bid-ask spread monotonically converges to

$$s_* = 2\sigma_v. \quad (19)$$

Thus, Proposition 4 shows that in the long run the equilibrium approaches a particular stationary equilibrium, which we analyze next.

4.2 Stationary Equilibrium

We define a *stationary equilibrium* an equilibrium in which the public volatility σ_t is constant. According to Proposition 4, if the initial density is $\phi_0(v) = \mathcal{N}(v, \mu_0, \sigma_*)$, then all subsequent

public densities have the same volatility, namely the stationary volatility σ_* . We now analyze the properties of the stationary equilibrium.

Corollary 2. *In the stationary equilibrium, the public volatility σ_* is decreasing in the fraction of informed trading ρ , while the bid-ask spread s_* does not depend on ρ . Both σ_* and s_* are increasing in the fundamental volatility σ_v .*

Intuitively, an increase in the fundamental volatility σ_v raises the public volatility as the dealer's knowledge about the fundamental value becomes more imprecise. It also increases the adverse selection overall for the dealer, hence she increases the bid-ask spread. Moreover, a decrease in the fraction of informed trading ρ means that the order flow becomes less informative, and therefore the dealer's knowledge about the fundamental value is more imprecise (σ_* is large).

The surprising result is that the stationary bid-ask spread is independent of ρ . This is equivalent to the public mean update being independent of ρ . Indeed, the public mean evolves according to

$$\mu_{t+1,B} = \mu_t + \sigma_v, \quad \mu_{t+1,S} = \mu_t - \sigma_v. \quad (20)$$

Thus, the bid-ask spread is $s_* = (\mu_t + \sigma_v) - (\mu_t - \sigma_v) = 2\sigma_v$. To understand the intuition behind this result, consider the case when ρ is low. Suppose the dealer observes a buy order at t . As ρ is low, there are two effects on the size of the public mean update. The first effect is negative: the trader at t is unlikely to be informed, which decreases the size of the update. This is the traditional *adverse selection effect* from models such as GM85. The second effect is positive: when the trader at t is informed, he must have observed a large fundamental value v_t , as the uncertainty in v_t (measured by the public volatility σ_*) is also large. This we call the *dynamic efficiency effect*: more informed traders create over time a more precise knowledge about the fundamental value, and thus reduce the effect of informational updates.

It turns out that the dynamic efficiency effect exactly cancels the adverse selection effect in a stationary setup, and as a result the magnitude of the public mean updates due to order flow is independent of ρ . To understand why, consider an equilibrium which is not necessarily stationary. If there was no order flow at t , then the dealer's uncertainty (the public volatility) would increase from t to $t + 1$ as the fundamental value diffuses. But there

is order flow at t , which provides information to the dealer and hence reduces uncertainty at $t+1$. In a stationary equilibrium the public uncertainty stays constant. Thus, as the increase in uncertainty due to value diffusion is independent of the informed share ρ , the decrease in uncertainty due to order flow should also be independent of ρ . But an order flow information content that is independent of ρ translates into the magnitude of public mean updates also being independent of ρ .

Formally, the decrease in uncertainty due to the order \mathcal{O}_t at t can be evaluated by comparing the prior public density $\phi_t(v)$ and the posterior density $\psi_t(v|\mathcal{O}_t)$. One measure of the decrease in uncertainty is how much the public mean is updated after a buy or sell order (which are equally likely). But (20) implies that this update is $\pm\sigma_v$, which from the point of view of the information at t is a binary distribution, with standard deviation σ_v which is indeed independent of ρ . Note that we have also essentially proved the following result.

Corollary 3. *In the stationary equilibrium, the volatility of the change in public mean is constant and equal to σ_v .*

This result is in fact true quite generally. Indeed, in Appendix B we prove that for any filtration problem in which the variance remains constant over time the volatility of the change in public mean must equal the fundamental volatility.

4.3 Liquidity Dynamics

In this section we analyze the evolution of the public volatility and the bid-ask spread after a shock to either the public volatility σ_t , the fundamental volatility σ_v , or the fraction of informed trading ρ . We are also interested in how quickly the equilibrium converges to the stationary equilibrium. In general, the speed of convergence of a sequence x_t that converges to a limit x_* is defined as the limit ratio

$$S = \lim_{t \rightarrow \infty} \frac{|x_t^2 - x_*^2|}{|x_{t+1}^2 - x_*^2|}, \quad (21)$$

provided that the limit exists. The next result computes the speed of convergence for several variables of interest.

Corollary 4. *The public volatility, public variance and bid-ask spread have the same speed of convergence:*

$$S = \frac{1}{1 - \delta^2}. \quad (22)$$

Moreover, S is increasing in the fraction of informed trading ρ .

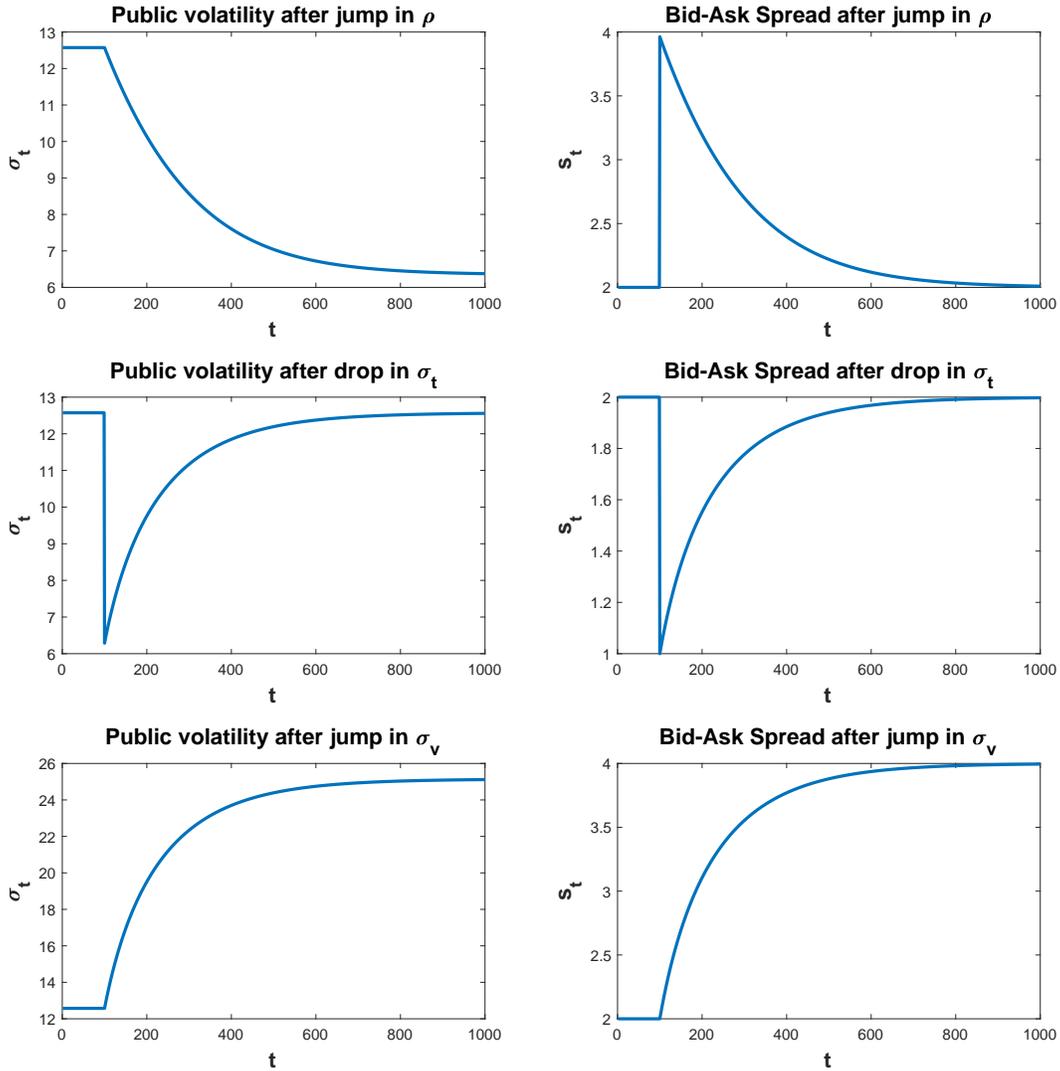
Corollary 4 shows that the variables of interest have the same speed of convergence S , and we can thus call S simply as the *convergence speed* of the equilibrium. Another result of Corollary 4 is that a larger fraction of informed trading ρ implies a faster convergence speed of the equilibrium to its stationary value. This is intuitive, as more informed trading helps the dealer make quicker dynamic inferences. Note that when $\rho = 1$, equation (15) implies that $\delta = g^{-1}(2) \approx 0.647$, thus the maximum value of δ is less than one. Therefore, the maximum convergence speed is finite.

We now consider the effect of various types of shocks to our stationary equilibrium. In the first row of Figure 1 we plot the effects of a positive shock to the fraction of informed trading, meaning that ρ suddenly jumps to a higher value ρ' . This generates an increase in δ , which jumps to its new value $\delta' = g^{-1}(\rho')$, and it also generates a drop in the stationary public volatility, which is now $\sigma'_* = \sigma_v/\delta'$. Nevertheless, as there is no new information above the fundamental value, the current public volatility σ_t remains equal to its old stationary value, $\sigma_* = \sigma_v/\delta$. Proposition 4 shows that the public volatility starts decreasing monotonically toward its stationary value σ'_* . Note that according to Corollary 4 the speed of convergence to the new stationary equilibrium is $S' = 1/(1 - \delta'^2)$, which is higher than the old convergence speed. We also describe the evolution of the bid-ask spread, which according to Proposition 3 satisfies $s_t = 2\delta'\sigma_t$. Initially, the bid-ask spread jumps to reflect the jump to δ' . But then, as σ_t converges to $\sigma'_* = \sigma_v/\delta'$, the bid-ask spread starts decreasing to $s_* = 2\sigma_v$, which does not depend on ρ .

To summarize, after a positive shock to ρ , the public volatility starts decreasing monotonically to its now lower stationary value, while the bid-ask spread initially jumps and then decreases monotonically to the same stationary value (that does not depend on ρ). Intuitively, a positive shock to the fraction of informed trading leads to a sudden increase in adverse selection for the dealer, reflected in an initially larger bid-ask spread, after which the bid-ask

Figure 1: Public Volatility and Bid-Ask Spread after Shocks.

This figure plots the effect of three types of shocks on the public volatility σ_t , and on the bid-ask spread s_t (each shock occurs at $t_0 = 100$). The initial parameters are: $\sigma_v = 1$, and $\rho = 0.1$ (hence $\delta = 0.0795$, $\sigma_* = 12.573$, $s_* = 2$). In the first row, the fraction of informed trading ρ jumps from 0.1 to 0.2 (hence σ_* drops from 12.573 to 6.345). In the second row, the public volatility drops from $\sigma_* = 12.573$ to half of its value (6.286). In the third row, the fundamental volatility jumps from 1 to 2.



spread reverts to its fundamental value, which is independent of informed trading. At the same time, more informed trading leads to more precision for the dealer in the long run, which is reflected in a smaller public volatility.

In the second row of Figure 1 we plot the effects of a negative shock to the public volatility, meaning that σ_t suddenly drops from the stationary value σ_* to a lower value. This drop can be caused for instance by public news about the value of the asset v_t . Then, according to Proposition 4, the public volatility increases monotonically back to the stationary value. The bid-ask spread is always proportional to the public density: $s_t = 2\delta\sigma_t$, hence s_t also drops initially and then increases monotonically toward the stationary value s_* . Intuitively, public news has the effect of helping the dealer initially to get a more precise understanding about the fundamental value. This brings down the bid-ask spread, as temporarily the dealer faces less adverse selection. But this decrease is only temporary, as the value diffuses and the same forces increase the public volatility and the bid-ask spread toward their corresponding stationary values, which are the same as before.

In the third row of Figure 1 we plot the effects of a positive shock to the fundamental volatility, meaning that σ_v suddenly jumps to a higher value σ'_v . This implies that every value increment $v_{t+1} - v_t$ now has higher volatility, but the uncertainty in v_t , which is measured by the public volatility σ_t , stays the same.¹¹ Proposition 4 shows that the stationary public volatility changes to $\sigma'_* = \sigma'_v/\delta$, and the stationary bid-ask spread changes to $s'_* = 2\sigma'_v$. Therefore, the public density increases monotonically from the initial stationary value to the new stationary value, and the same is true for the bid-ask spread. Intuitively, a larger fundamental volatility increases overall adverse selection for the dealer, and as a result both the public density and the bid-ask spread eventually increase.

5 Equilibrium with Exact Bayesian Inference

In this section, we analyze in more detail the evolution of the public density ϕ_t when the dealer is fully Bayesian. In particular, we are interested in computing the average shape of the public density over all possible future paths of the game.¹² Note that when computing the average

¹¹One can mix this type of shock with a shock to the public volatility σ_t , which was already analyzed.

¹²As we see in Internet Appendix (Sections 1 and 2), one expects a well defined stationary density for the continuous time Markov chain associated to our game. The only problem is that the fundamental value v_t is no longer stationary in our case, but follows a random walk. One can show that it is still possible to define a stationary density as long as one does not require it to integrate to one over v . But we are interested in the simpler problem of computing the marginal stationary density of $v_t - \mu_t$, which we solve numerically.

shape of a density, we consider the average of the various densities after demeaning them. We then show numerically that this average exists and is not too far from the stationary public density described in Section 4.2, which is a normal density with mean zero and standard deviation equal to $\sigma_* = \sigma_v/\delta$.

We thus demean the variables and densities involved in the previous formulas, and in addition we normalize them by σ_* :

$$\begin{aligned} \tilde{A}_t &= \frac{A_t - \mu_t}{\sigma_*}, & \tilde{B}_t &= \frac{B_t - \mu_t}{\sigma_*}, & \tilde{v}_t &= \frac{v_t - \mu_t}{\sigma_*}, & \tilde{\phi}_t(\tilde{v}) &= \sigma_* \phi_t(\mu_t + \sigma_* \tilde{v}), \\ \tilde{\Phi}_t(\tilde{v}) &= \int_{-\infty}^{\tilde{v}} \tilde{\phi}_t(w) dw, & \tilde{\psi}_t(\tilde{v}|\mathcal{O}_t) &= \sigma_* \psi_t(\mu_t + \sigma_* \tilde{v}|\mathcal{O}_t). \end{aligned} \quad (23)$$

With this new notation, the equations (4) and (5) from Proposition 1 imply the following result.

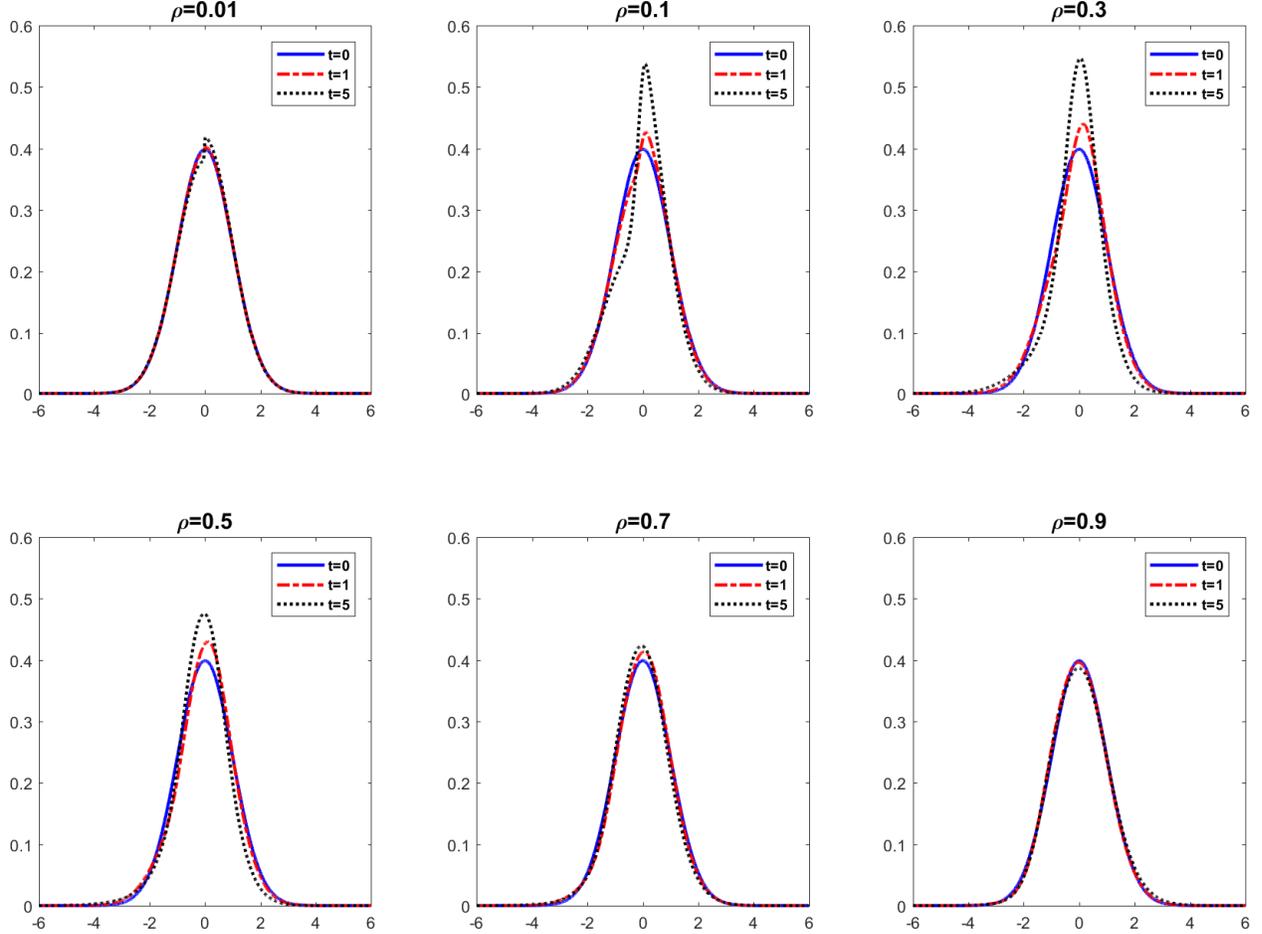
Corollary 5. *Consider a rapidly decaying public density ϕ_t with normalization $\tilde{\phi}_t$, and an ask-bid pair (A_t, B_t) with normalization $(\tilde{A}_t, \tilde{B}_t)$. After observing an order $\mathcal{O}_t \in \{\text{B}, \text{S}\}$, the normalized density at $t+1$ is $\tilde{\phi}_{t+1}(\tilde{w}|\mathcal{O}_t)$, where*

$$\begin{aligned} \tilde{\phi}_{t+1}(\tilde{w}|\text{B}) &= \int_{-\infty}^{+\infty} \mathcal{N}\left(\frac{\tilde{w} - \tilde{v} + \tilde{A}_t}{\delta}\right) \frac{(\rho \mathbf{1}_{\tilde{v} > \tilde{A}_t} + \frac{\rho}{2} \mathbf{1}_{\tilde{v} \in [\tilde{B}_t, \tilde{A}_t]} + \frac{1-\rho}{2}) \cdot \tilde{\phi}_t(\tilde{v})}{\frac{\rho}{2}(1 - \tilde{\Phi}_t(\tilde{A}_t)) + \frac{\rho}{2}(1 - \tilde{\Phi}_t(\tilde{B}_t)) + \frac{1-\rho}{2}} d\tilde{v}, \\ \tilde{\phi}_{t+1}(\tilde{w}|\text{S}) &= \int_{-\infty}^{+\infty} \mathcal{N}\left(\frac{\tilde{w} - \tilde{v} + \tilde{B}_t}{\delta}\right) \frac{(\rho \mathbf{1}_{\tilde{v} < \tilde{B}_t} + \frac{\rho}{2} \mathbf{1}_{\tilde{v} \in [\tilde{B}_t, \tilde{A}_t]} + \frac{1-\rho}{2}) \cdot \tilde{\phi}_t(\tilde{v})}{\frac{\rho}{2}\tilde{\Phi}_t(\tilde{A}_t) + \frac{\rho}{2}\tilde{\Phi}_t(\tilde{B}_t) + \frac{1-\rho}{2}} d\tilde{v}. \end{aligned} \quad (24)$$

Figure 2 displays the normalized public density after $t = 0$, $t = 1$, and $t = 5$ buy orders for various values of the informed share ρ . We notice by visual inspection that the normalized public density is close to the standard normal density even after a sequence of 5 buy orders (this sequence happens with probability 2^{-5} , which is approximately 3.13%). The deviation of the normalized public densities from the standard normal density is at its smallest level when the fraction of informed trading ρ is either small or large, and it peaks for an intermediate value ρ near 0.2. When ρ is small, the order flow is uninformative, hence the posterior is not far from the prior. When ρ is large, the order flow is very informative, hence the posterior depends strongly on the increment, which is normally distributed.

Figure 2: Exact Normalized Public Density after Series of Buy Orders.

Each of the 6 plots represents the evolution of the normalized public density $\tilde{\phi}_t$ after $t = 0$, $t = 1$ and $t = 5$ buy orders. The initial normalized public density in all cases (at $t = 0$) is the standard normal density with mean zero and volatility one. The 6 plots correspond to the fraction of informed trading $\rho \in \{0.01, 0.1, 0.3, 0.5, 0.7, 0.9\}$.



It turns out, however, that the stationary shape of the public density is not precisely normal, but it has “fat tails,” that is, its fourth centralized moment (kurtosis) is larger than 3. Table 1 displays, for each $\rho \in \{0.01, 0.1, 0.3, 0.5, 0.7, 0.9\}$, several moments of the average normalized density computed after 200 different random paths. As the starting density (at $t = 0$) is standard normal for all the different ρ , we need to make sure that we choose a path length long enough for the average density to stabilize. Numerically, we see that it is enough

Table 1: Average Normalized Public Density after Series of Random Orders.

For each informed share $\rho \in \{0.01, 0.1, 0.3, 0.5, 0.7, 0.9\}$, consider 200 random series of 20 orders chosen among buy or sell with equal probability, and denote by $\tilde{\phi}_S$ the normalized public density computed after observing the series $S = 1, 2, \dots, 200$. The table displays four estimated moments of the average $\psi = \frac{\tilde{\phi}_1 + \tilde{\phi}_2 + \dots + \tilde{\phi}_{200}}{200}$: the mean $\mu = \int_{-\infty}^{+\infty} x\psi(x)dx$, the standard deviation $\sigma = (\int_{-\infty}^{+\infty} (x-\mu)^2\psi(x)dx)^{1/2}$, the skewness $\int_{-\infty}^{+\infty} (\frac{x-\mu}{\sigma})^3\psi(x)dx$, and the kurtosis $\int_{-\infty}^{+\infty} (\frac{x-\mu}{\sigma})^4\psi(x)dx$. It also displays the average bid-ask spread normalized by $s_* = 2\sigma_v$ (N.Spread).

ρ	0.01	0.1	0.3	0.5	0.7	0.9
Mean	-0.000	0.000	-0.001	-0.001	-0.010	-0.002
St.Dev.	1.000	1.002	1.039	1.056	1.041	1.009
Skewness	-0.001	0.018	0.016	-0.003	-0.012	-0.009
Kurtosis	3.005	3.419	4.587	4.597	4.089	3.343
N.Spread	1.003	0.966	0.959	0.988	1.014	1.004

to choose $t = 20$.¹³ Thus, in Table 1 we display the first four centralized moments for the average normalized public density at $t = 20$, computed over 200 random paths.

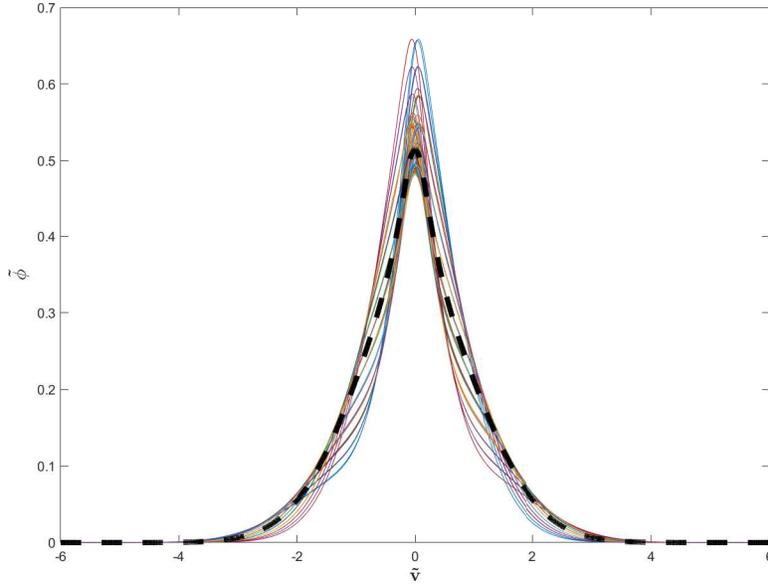
The first three moments of the average density at $t = 20$ are similar to the moments of the standard normal density: the mean and the skewness (centralized third moment) are close to zero, and the standard deviation is close to one. The kurtosis, however, is larger than 3, indicating that the stationary public density has indeed fat tails. Nevertheless, the deviation from the standard normal density is not large, especially when ρ is small or large. Moreover, the last row in Table 1 implies that the average bid-ask spread in each case is quite close to $s_* = 2\sigma_v$, which is the stationary value in the approximate Bayesian case: see equation (19). Thus, we argue that the normal approximation made in Section 4 is reasonable, especially when it comes to our main liquidity measure, the bid-ask spread.

The question remains how different the normalized public density can be from the *average* density. This question is already discussed tangentially in Figure 2, where we observe the normalized public density after five buy orders. But to understand this issue in more detail,

¹³We have checked that the average density at $t = 20$ is in absolute value less than 0.01 apart from the average density at $t = 25$ or $t = 30$.

Figure 3: Normalized Public Density after Series of Random Orders.

For an informed share $\rho = 0.1$, consider 200 random series of 20 orders chosen among buy or sell with equal probability, and denote by $\tilde{\phi}_S$ the normalized public density computed after observing the series $S = 1, 2, \dots, 200$. The table displays the densities $\tilde{\phi}_S$, as well as their average $\psi = \frac{\tilde{\phi}_1 + \tilde{\phi}_2 + \dots + \tilde{\phi}_{200}}{200}$. The average density is displayed with a thick dashed line.



we choose one particular value of the informed share, $\rho = 0.1$, for which the normalized public density after five buy orders appears more different than the normal density. Figure 3 displays the normalized public density after each of the 200 random series of 20 orders, along with the average density. Then, the results in Table 1 and Figure 3 can be summarized by observing that the normalized public density does not deviate too far from its average value, and in turn this average value does not deviate too far from the standard normal density.

6 Robustness and Extensions

6.1 Discussion

In this section we discuss whether our main result (that liquidity is not affected by the fraction ρ of informed trading) remains true if we modify the model assumptions. For that, we recall the intuition behind that result, as described in Section 4.2. Consider the case when ρ is low,

and suppose the dealer observes a buy order at t . First, we have the *adverse selection effect*: a low ρ means that the buyer is unlikely to be informed, which implies that the update of public mean (and hence the dealer's bid-ask spread) should be small. Second, there is an opposing *dynamic efficiency effect*: in the rare case when the buyer is actually informed, he must have observed a large fundamental value v_t , as the uncertainty in v_t (measured by the public volatility σ_*) is also large.

Note that for this offsetting argument to fully work, the public volatility σ_* must be very large when ρ is very small. This is possible only if the range of the fundamental value is not restricted to become very large. Such restrictions can occur in two ways: either (i) the fundamental value is directly assumed to be bounded, or (ii) the signals received by the dealer essentially bound the dealer's uncertainty.

Situation (i) occurs if we require the fundamental value to lie in a bounded interval such as $[0, 1]$.¹⁴ We analyze such a model in the Internet Appendix Section 2, where the fundamental value is either zero or one (as in Glosten and Milgrom, 1985), and it switches every period between these two values with probability $\nu < 1/2$. In that case, the dynamic efficiency effect no longer offsets the adverse selection effect. Nevertheless, even if ρ is small, as long as ρ is large relative to the switching parameter ν , the dynamic efficiency effect is relatively strong, and as a result the dependence of the average bid-ask spread on ρ is weaker, and the equilibrium approaches the one in the diffusing-value model where the average bid-ask spread is independent of ρ .

Situation (ii) occurs if the dealer receives at every t signals about the level v_t . Note that this is not by itself enough to bound dealer's uncertainty about v_t . Indeed, in Section 6.2 we consider an extension of our model in which, in addition to observing the order flow, the dealer receives at every t signals about the increment $v_t - v_{t-1}$. In that case, we see that the main result goes through. The reason is that the dealer's uncertainty about the level v_t becomes large when ρ is small: the dealer only learns about v_t once, after which she learns only about future value increments. If, by contrast, the dealer received at every t signals about the level v_t , then the uncertainty would remain bounded even when ρ is very small,

¹⁴This setup of course cannot occur if the fundamental value follows a random walk.

and the main result would no longer hold.

6.2 Public News

We now analyze an extension of the model in Section 2, in which the dealer receives news every period (this could be interpreted as the dealer receiving public news). Specifically, suppose that before each $t = 1, 2, \dots$ the dealer receives a signal $\Delta s_t = s_t - s_{t-1}$ about the increment $\Delta v_t = v_t - v_{t-1}$:¹⁵

$$\Delta s_t = \Delta v_t + \Delta \eta_t, \quad \text{with} \quad \Delta \eta_t = \eta_t - \eta_{t-1} \stackrel{IID}{\sim} \mathcal{N}(\cdot, 0, \sigma_\eta). \quad (25)$$

Denote, respectively, by μ_t and σ_t the public mean and public volatility just before trading at $t = 0, 1, 2, \dots$ (but after the signal Δs_t is observed). Note that this extension generalizes the model in Section 2: when σ_η approaches infinity, it is as if the dealer receives no signal at t . The next result generalizes Proposition 4.

Proposition 5. *For any $t = 0, 1, 2, \dots$ the public mean and volatility satisfy*

$$\mu_{t+1} = \mu_t \pm \delta \sigma_t + \frac{\sigma_v^2}{\sigma_v^2 + \sigma_\eta^2} \Delta s_{t+1}, \quad \sigma_t^2 = \sigma_*^2 + (\sigma_0^2 - \sigma_*^2) (1 - \delta^2)^t, \quad (26)$$

where $\delta = g^{-1}(2\rho)$, as in equation (15), and

$$\sigma_* = \frac{\sigma_{v\eta}}{\delta} \quad \text{with} \quad \sigma_{v\eta} = \frac{\sigma_v \sigma_\eta}{\sqrt{\sigma_v^2 + \sigma_\eta^2}}. \quad (27)$$

For any initial value σ_0 and any sequence of orders, the public volatility σ_t monotonically converges to σ_ , and the bid-ask spread monotonically converges to*

$$s_* = 2\sigma_{v\eta}. \quad (28)$$

¹⁵Alternatively, but perhaps less realistically, the dealer receives in each period t a signal about the level v_t . In this case, there is an upper bound for the dealer's uncertainty even if the informed share is very small. Hence, the public volatility is no longer increasing indefinitely with the informed share, and therefore the dynamic efficiency effect is reduced. As a result, the adverse selection effect dominates the dynamic efficiency effect, and the stationary bid-ask spread is increasing in the informed share.

Proposition 5 is essentially the same result as Proposition 4, except that the fundamental volatility σ_v is replaced here by $\sigma_{v\eta}$. The parameter $\sigma_{v\eta}$ represents the increase in dealer uncertainty from t to $t + 1$, conditional on her receiving the signal Δs_{t+1} .¹⁶ When σ_η is zero, the dealer learns perfectly the increment Δv , hence even if the dealer does not know the initial value v_0 , she ends up by learning v_t almost perfectly (she also learns about v_t from the order flow). When σ_η approaches infinity, the dealer receives uninformative signals, $\sigma_{v\eta}$ approaches σ_v , and the equilibrium behavior is described as in Proposition 4.

The stationary bid-ask spread s_* is twice the parameter $\sigma_{v\eta}$. Thus, the bid-ask spread is increasing in the news uncertainty parameter σ_η , and ranges from zero (when $\sigma_\eta = 0$) to $2\sigma_v$ (when $\sigma_\eta = \infty$). The relation between the bid-ask spread and σ_η is intuitive: with more imprecise news, the dealer is more uncertain about the asset value, and sets a larger stationary bid-ask spread.

Note that even in this more general context the stationary bid-ask spread s_* does not depend on the informed share ρ . The intuition is the same as for Proposition 4, and is discussed at the end of the proof of Proposition 5. This intuition is based on the general result (proved in Appendix B) that for any filtration problem in which the variance remains constant over time, the variance of the change in public mean must equal the fundamental variance. But the latter variance is independent of ρ , as is the variance of the signal Δs , hence the bid-ask spread is also independent of ρ .

7 Conclusion

In this paper we have presented a dealer model in which the asset value follows a random walk. The stationary equilibrium of the model has novel properties. Our main finding is that the stationary bid-ask spread no longer depends on the informed share (the fraction of traders that are informed). This result is driven by two offsetting effects: (i) the traditional adverse selection effect: the dealer sets higher bid-ask spreads to protect from a larger number of informed traders, and (ii) the dynamic efficiency effect: the dealer learns faster from the order flow when there are more informed traders, and this reduces the bid-ask spread.

¹⁶Indeed, its square $\sigma_{v\eta}^2$ is equal to the conditional variance $\text{Var}(\Delta v_{t+1} | \Delta s_{t+1})$.

The non-stationary equilibria converge to the stationary equilibrium, regardless of the initial state. The evolution of the non-stationary equilibrium after various types of shocks provides additional testable implications of our model. For instance, after a positive shock to the informed share (e.g., if more informed investors start trading in that stock) the bid-ask spread jumps but then it decreases again to its stationary level. This type of liquidity resilience occurs purely for informational reasons, without any additional market maker jumping in to provide liquidity.

Appendix A. Proofs of Results

Proof of Proposition 1. Using Bayes' rule, the posterior density of v_t after observing \mathcal{O} is

$$\psi_t(v|\mathcal{O}) = \frac{\mathbf{P}(\mathcal{O}_t = \mathcal{O} \mid v_t = v) \cdot \mathbf{P}(v_t = v)}{\int_v \mathbf{P}(\mathcal{O}_t = \mathcal{O} \mid v_t = v) \cdot \mathbf{P}(v_t = v)} = \frac{g_t(\mathcal{O}, v) \cdot \phi_t(v)}{\int_v g_t(\mathcal{O}, v) \cdot \phi_t(v)}, \quad (\text{A1})$$

where $\int_v F(v)$ is shorthand for $\int_{-\infty}^{+\infty} F(v)dv$. Substituting $g_t(\mathcal{O}, v)$ from (2) and (3) in the above equation, we obtain (4).

Let $f(w, v) = \mathbf{P}(v_{t+1} = w \mid v_t = v) = \mathcal{N}(w - v, 0, \sigma_v)$ be the transition density of v_t . To compute the posterior density of v_t after observing $\mathcal{O}_t = \mathcal{O}$, note that

$$\begin{aligned} \phi_{t+1}(w|\mathcal{O}) &= \int_v \mathbf{P}(v_{t+1} = w \mid v_t = v, \mathcal{O}_t = \mathcal{O}) \cdot \mathbf{P}(v_t = v \mid \mathcal{O}_t = \mathcal{O}) \\ &= \int_v \mathbf{P}(v_{t+1} = w \mid v_t = v) \cdot \mathbf{P}(v_t = v \mid \mathcal{O}_t = \mathcal{O}) = \int_v f(w, v) \cdot \psi_t(v|\mathcal{O}), \end{aligned} \quad (\text{A2})$$

which proves (5).

To simplify notation, we omit conditioning on the order \mathcal{O}_t . From (4), it follows that the posterior density ψ_t is equal to ϕ_t multiplied by a piecewise constant function. The prior density ϕ_t is rapidly decaying, hence it is bounded. Therefore ψ_t is also bounded and continuous, although it is no longer smooth. Nevertheless, when we convolute $\psi_t(\cdot)$ with $\mathcal{N}(\cdot, 0, \sigma_v)$ the result ϕ_{t+1} becomes smooth. Indeed, the N 'th derivative $d^N \phi_{t+1}(w)/dw^N$ involves differentiating the smooth function $\mathcal{N}(w - v, 0, \sigma_v)$ under the integral sign. As the

remaining term $\psi_t(v)$ is bounded, the integrals are well defined, and hence ϕ_{t+1} is a smooth function. The fact that ϕ_{t+1} is also rapidly decaying can be seen in the same way, using again the fact that ψ_t is bounded. \square

Proof of Corollary 1. By definition of the ask-bid pair, A_t is the mean of the posterior density of v_t after observing a buy order at t . But the increment $v_{t+1} - v_t$ has zero mean and is independent of the previous variables until t . Therefore, A_t is also the mean of the posterior density of v_{t+1} after observing a buy order at t . Similarly, B_t is the mean of the posterior density of v_{t+1} after observing a sell order at t . This proves the equations in (7). \square

Proof of Proposition 2. Define the following function:¹⁷

$$H_t(v) = \int_{-\infty}^v w\phi_t(w)dw = v\Phi_t(v) - \int_{-\infty}^v \Phi_t(w)dw. \quad (\text{A3})$$

Note that $H_t(-\infty) = 0$ and $H_t(+\infty) = \int_{-\infty}^{\infty} w\phi_t(w)dw = \mu_t$. Also, note that

$$\Theta_t(v) = \mu_t\Phi_t(v) - H_t(v). \quad (\text{A4})$$

To prove the desired equivalence, start with an ask-bid pair (A_t, B_t) . This pair must satisfy the dealer's pricing conditions: A_t is the mean of $\psi_t(\cdot|\text{B})$, and B_t is the mean of $\psi_t(\cdot|\text{S})$. Using the formulas in (4) for $\psi_t(v|\mathcal{O})$, we compute

$$\begin{aligned} A_t &= \frac{\rho(\mu_t - H_t(A_t)) + \frac{\rho}{2}(H_t(A_t) - H_t(B_t)) + \frac{1-\rho}{2}\mu_t}{\frac{\rho}{2}(1 - \Phi_t(A_t)) + \frac{\rho}{2}(1 - \Phi_t(B_t)) + \frac{1-\rho}{2}}, \\ B_t &= \frac{\rho H_t(B_t) + \frac{\rho}{2}(H_t(A_t) - H_t(B_t)) + \frac{1-\rho}{2}\mu_t}{\frac{\rho}{2}\Phi_t(A_t) + \frac{\rho}{2}\Phi_t(B_t) + \frac{1-\rho}{2}}. \end{aligned} \quad (\text{A5})$$

¹⁷In the formula for H_t we use integration by parts, and also the fact that $\lim_{v \rightarrow -\infty} v\Phi_t(v) = 0$. To prove this last fact, suppose $v = -x$ with $x > 0$. Since ϕ_t is rapidly decaying, $\phi_t(-x) < Cx^{-3}$ for some constant C . Then $x\Phi_t(-x) = x \int_{-\infty}^{-x} \phi_t(w)dw < x \frac{Cx^{-2}}{2}$, which implies $\lim_{x \rightarrow \infty} x\Phi_t(-x) = 0$.

Using (A4), we compute the following differences:

$$\begin{aligned} A_t - \mu_t &= \frac{\frac{\rho}{2}\Theta_t(A_t) + \frac{\rho}{2}\Theta_t(B_t)}{\rho(1 - \Phi_t(A_t)) + \frac{\rho}{2}(\Phi_t(A_t) - \Phi_t(B_t)) + \frac{1-\rho}{2}}, \\ \mu_t - B_t &= \frac{\frac{\rho}{2}\Theta_t(A_t) + \frac{\rho}{2}\Theta_t(B_t)}{\rho\Phi_t(B_t) + \frac{\rho}{2}(\Phi_t(A_t) - \Phi_t(B_t)) + \frac{1-\rho}{2}}. \end{aligned} \quad (\text{A6})$$

As Θ_t is strictly positive everywhere (see Footnote 9), we have the following inequalities: $A_t > \mu_t > B_t$, or equivalently $A_t \in (\mu_t, +\infty)$ and $B_t \in (-\infty, \mu_t)$. The equations (A6) can be written as

$$F(A_t, B_t) = 0, \quad G(A_t, B_t) = 0, \quad (\text{A7})$$

where the functions F and G are defined in (8). Conversely, suppose we have a solution (A_t, B_t) of (A7), with $A_t > \mu_t > B_t$. Then, this pair satisfies the equations in (A6), which are the dealer's pricing conditions. Thus, (A_t, B_t) is an ask-bid pair.

We now show that a solution of (A7) exists. The partial derivatives of F and G are

$$\begin{aligned} \frac{\partial F}{\partial A} &= -\frac{\Theta_t(A) + \Theta_t(B)}{(A - \mu_t)^2}, & \frac{\partial F}{\partial B} &= \frac{A - B}{A - \mu_t} \phi_t(B), \\ \frac{\partial G}{\partial A} &= -\frac{A - B}{\mu_t - B} \phi_t(A), & \frac{\partial G}{\partial B} &= \frac{\Theta_t(A) + \Theta_t(B)}{(\mu_t - B)^2}. \end{aligned} \quad (\text{A8})$$

From (8) we see that $F(A, B)$ has well defined limits at $B = \pm\infty$, which follows from the formulas: $\Theta_t(\pm\infty) = 0$, $\Phi_t(-\infty) = 0$, and $\Phi_t(+\infty) = 1$. Thus we extend the definition of F for all $B \in \bar{\mathbb{R}} = [-\infty, +\infty]$. Now fix $B \in \bar{\mathbb{R}}$. We show that there is a unique solution $A = \alpha(B)$ of the equation $F(A, B) = 0$. From (A8) we see that $\frac{\partial F}{\partial A} < 0$ for all $A \in (\mu_t, \infty)$. From (8) we see that when $A \searrow \mu_t$, $F(A, B) \nearrow \infty$; while when $A \nearrow \infty$, $F(A, B) \searrow -\frac{1+\rho}{\rho} + 1 + \Phi_t(B) = -\frac{1}{\rho} + \Phi_t(B) < 0$ (recall that $\rho \in (0, 1)$). Thus, for any B there is a unique solution of $F(A, B) = 0$ for $A \in (\mu_t, \infty)$. Denote this unique solution by $\alpha(B)$. Differentiating the equation $F(\alpha(B), B) = 0$ implies that for all B the derivative of $\alpha(B)$ is $\alpha'(B) = -\frac{\partial F}{\partial B}(\alpha(B), B) / \frac{\partial F}{\partial A}(\alpha(B), B) > 0$. Define $\underline{A} = \alpha(-\infty)$ and $\bar{A} = \alpha(\mu_t)$. The results above imply that both \underline{A} and \bar{A} belong to (μ_t, ∞) , and α is a bijective function between $[-\infty, \mu_t]$ and $[\underline{A}, \bar{A}]$.

A similar analysis shows that for all $A \in \bar{\mathbb{R}}$, there is a unique solution $B = \beta(A)$ of the

equation $G(A, B) = 0$. Moreover, the function β is increasing, and if we define $\underline{B} = \beta(\mu_t)$ and $\overline{B} = \beta(\infty)$, it follows that both \underline{B} and \overline{B} belong to $(-\infty, \mu_t)$, and the function α is bijective between $[\mu_t, \infty]$ and $[\underline{B}, \overline{B}]$.

Next, define the function $f : \overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}}$ by

$$f(A) = \alpha(\beta(A)). \quad (\text{A9})$$

Consider the set

$$S = \{(A, B) \mid A - f(A) = 0, B = \beta(A)\}. \quad (\text{A10})$$

It is straightforward to show that S coincides with the set of all ask-bid pairs. Indeed, $(A, B) \in S$ is equivalent to $A = \alpha(B)$ and $B = \beta(A)$, which, from the discussion above, is equivalent to $F(A, B) = 0$ and $G(A, B) = 0$. Therefore, the existence of an ask-bid pair is equivalent to there being at least one solution of $A - f(A) = 0$.

We now show that the equation $A - f(A) = 0$ has at least one solution. The function $f(A)$ is increasing and bijective between $[\mu_t, \infty]$ and $[\alpha(\underline{B}), \alpha(\overline{B})]$. As $\underline{B}, \overline{B} \in (-\infty, \mu_t)$, it follows that $[\alpha(\underline{B}), \alpha(\overline{B})] \subset (\underline{A}, \overline{A}) \subset (\mu_t, \infty)$. When $A \searrow \mu_t$, $A - f(A) \rightarrow \mu_t - \alpha(\underline{B}) < 0$, while when $A \nearrow \infty$, $A - f(A) \rightarrow \infty - \alpha(\overline{B}) > 0$. Thus, there exists a solution of $A - f(A) = 0$ on (μ_t, ∞) .

Finally, define

$$A_t = \inf\left\{A \in (\mu_t, \infty) \mid A - f(A) = 0\right\}, \quad B_t = \beta(A_t). \quad (\text{A11})$$

As f is continuous, A_t also satisfies $A_t - f(A_t) = 0$, hence among all possible ask-bid pairs the ask closest to μ_t is attained at A_t . \square

Proof of Proposition 3. Denote by $\phi(\cdot) = \mathcal{N}(\cdot, 0, 1)$ the standard normal density, and by $\Phi(\cdot)$ its cumulative density. Recall that $\phi_t(\cdot)$ is the density of v_t just before trading at t , and $\psi_t(\cdot \mid \mathcal{O}_t)$ is the density of v_t after trading at t . In this proposition, we assume that we start with a normal density

$$\phi_t(v) = \frac{1}{\sigma_t} \phi\left(\frac{v - \mu_t}{\sigma_t}\right), \quad (\text{A12})$$

with mean μ_t and volatility σ_t . Define the normalized ask and bid, respectively, by

$$a_t = \frac{A_t - \mu_t}{\sigma_t}, \quad b_t = \frac{B_t - \mu_t}{\sigma_t}. \quad (\text{A13})$$

We now compute the mean and volatility of $\phi_{t+1}(\cdot|\mathcal{O}_t)$. As the increment $v_{t+1} - v_t \sim \mathcal{N}(0, \sigma_v^2)$ is independent of past variables, the mean and volatility of $\phi_{t+1}(\cdot|\mathcal{O}_t)$ satisfy

$$\mu_{t+1, \mathcal{O}_t} = \int_v v \psi_t(v|\mathcal{O}_t), \quad \sigma_{t+1, \mathcal{O}_t}^2 = \sigma_v^2 + \int_v (v - \mu_{t+1, \mathcal{O}_t})^2 \psi_t(v|\mathcal{O}_t). \quad (\text{A14})$$

From (4), we have $\psi_t(v|B) = \frac{\rho \mathbf{1}_{v > A_t} + \frac{\rho}{2} \mathbf{1}_{v \in [B_t, A_t]} + \frac{1-\rho}{2}}{\frac{\rho}{2}(1-\Phi_t(A_t)) + \frac{\rho}{2}(1-\Phi_t(B_t)) + \frac{1-\rho}{2}} \phi_t(v)$. With the change of variables $z = \frac{v-\mu_t}{\sigma_t}$, we compute posterior mean conditional on a buy order:

$$\begin{aligned} \mu_{t+1, B} &= \mu_t + \int_{-\infty}^{+\infty} (v - \mu_t) \frac{\rho \mathbf{1}_{v > A_t} + \frac{\rho}{2} \mathbf{1}_{v \in [B_t, A_t]} + \frac{1-\rho}{2}}{\frac{\rho}{2}(1-\Phi_t(A_t)) + \frac{\rho}{2}(1-\Phi_t(B_t)) + \frac{1-\rho}{2}} \frac{1}{\sigma_t} \phi\left(\frac{v-\mu_t}{\sigma_t}\right) dv \\ &= \mu_t + \sigma_t \int_{-\infty}^{+\infty} z \frac{\rho \mathbf{1}_{z > a_t} + \frac{\rho}{2} \mathbf{1}_{z \in [b_t, a_t]} + \frac{1-\rho}{2}}{\frac{\rho}{2}(1-\Phi(a_t)) + \frac{\rho}{2}(1-\Phi(b_t)) + \frac{1-\rho}{2}} \phi(z) dz \\ &= \mu_t + \sigma_t \frac{\phi(-a_t) + \phi(-b_t)}{\Phi(-a_t) + \Phi(-b_t) + \frac{1-\rho}{\rho}}. \end{aligned} \quad (\text{A15})$$

Similarly, the posterior mean conditional on a sell order is

$$\mu_{t+1, S} = \mu_t - \sigma_t \frac{\phi(a_t) + \phi(b_t)}{\Phi(a_t) + \Phi(b_t) + \frac{1-\rho}{\rho}}, \quad (\text{A16})$$

To compute $\sigma_{t+1, \mathcal{O}_t}^2$, we notice that

$$\int_v (v - \mu_t)^2 \psi_t(v|\mathcal{O}_t) = \int_v (v - \mu_{t+1, \mathcal{O}_t})^2 \psi_t(v|\mathcal{O}_t) + (\mu_{t+1, \mathcal{O}_t} - \mu_t)^2, \quad (\text{A17})$$

where we use the fact that $\int_v (v - \mu_{t+1, \mathcal{O}_t}) \psi_t(v|\mathcal{O}_t) = 0$. Using (A14) and (A17), a similar

calculation as in (A15) implies that the posterior variance conditional on a buy order satisfies

$$\begin{aligned}
\sigma_{t+1,B}^2 - \sigma_v^2 + (\mu_{t+1,B} - \mu_t)^2 &= \int_v (v - \mu_t)^2 \psi_t(v|B) \\
&= \sigma_t^2 \int_{-\infty}^{+\infty} z^2 \frac{\rho \mathbf{1}_{z>a_t} + \frac{\rho}{2} \mathbf{1}_{z \in [b_t, a_t]} + \frac{1-\rho}{2}}{\frac{\rho}{2}(1 - \Phi(a_t)) + \frac{\rho}{2}(1 - \Phi(b_t)) + \frac{1-\rho}{2}} \phi(z) dz \\
&= \sigma_t^2 \left(1 + \frac{a_t \phi(a_t) + b_t \phi(b_t)}{\Phi(-a_t) + \Phi(-b_t) + \frac{1-\rho}{\rho}} \right). \tag{A18}
\end{aligned}$$

Similarly, the posterior variance conditional on a sell order satisfies

$$\sigma_{t+1,S}^2 - \sigma_v^2 + (\mu_{t+1,S} - \mu_t)^2 = \sigma_t^2 \left(1 - \frac{a_t \phi(a_t) + b_t \phi(b_t)}{\Phi(a_t) + \Phi(b_t) + \frac{1-\rho}{\rho}} \right). \tag{A19}$$

We now use the fact that a_t and b_t are the normalized ask and bid. Equation (7) implies that the ask is $A_t = \mu_{t+1,B}$ and the bid is $B_t = \mu_{t+1,S}$. If we normalize these equations, we have $a_t = \frac{\mu_{t+1,B} - \mu_t}{\sigma_t}$ and $b_t = \frac{\mu_{t+1,S} - \mu_t}{\sigma_t}$. Using (A15) and (A16), we obtain

$$a_t = \frac{\phi(-a_t) + \phi(-b_t)}{\Phi(-a_t) + \Phi(-b_t) + \frac{1-\rho}{\rho}}, \quad b_t = -\frac{\phi(a_t) + \phi(b_t)}{\Phi(a_t) + \Phi(b_t) + \frac{1-\rho}{\rho}}, \tag{A20}$$

We show that this system has a unique solution. We use the notation from the proof of Proposition 2, adapted to this particular case. For $(a, b) \in (0, \infty) \times (-\infty, 0)$, define $F(a, b) = \frac{\phi(a) + \phi(b)}{a} - \Phi(-a) - \Phi(-b) - \frac{1-\rho}{\rho}$ and $G(a, b) = \frac{\phi(a) + \phi(b)}{-b} - \Phi(a) - \Phi(b) - \frac{1-\rho}{\rho}$. As in the proof of Proposition 2, for $b \in [-\infty, 0]$ define $\alpha(b)$ as the unique solution of $F(\alpha(b), b) = 0$; and for $a \in [0, \infty]$ define $\beta(a)$ as the unique solution of $G(a, \beta(a)) = 0$. For $a \in (0, \infty)$, define $f(a) = \alpha(\beta(a))$. We show that any solution a of the equation $a - f(a) = 0$ must satisfy $a < 1$. Let $b = \beta(a)$. Since $a = \alpha(b)$, by definition $F(a, b) = 0$. As in the proof of Proposition 2, one shows that F is decreasing in a , and that $F(0, b) = +\infty > 0$. As $b < 0$, $F(1, b) = \phi(1) - \Phi(-1) + \phi(b) - \Phi(-b) - \frac{1-\rho}{\rho} < \phi(1) - \Phi(-1) + \phi(0) - \Phi(0) \approx -0.0177 < 0$. As $F(a, b) = 0$ and F is decreasing in a (and a is positive), we have just proved that $a \in (0, 1)$. A similar argument (adapted to the function G) shows that $b = \beta(a) \in (-1, 0)$.

As in equation (A9), define $f(a) = \alpha(\beta(a))$. We need to show that the equation $a - f(a) = 0$ has a unique solution in $(0, \infty)$. By contradiction, suppose there are at least two solutions

$a_1 < a_2$, and suppose a_1 is the smallest such solution and a_2 the largest. As f is continuous and takes values in some compact interval $[\underline{b}, \bar{b}]$ (see the proof of Proposition 2), a_1 and a_2 are well defined. Also, since f is increasing, f is a bijection of $[a_1, a_2]$. The argument above then shows that both a_1 and a_2 are in $(0, 1)$. If we prove that $f' < 1$ on $[a_1, a_2]$, it follows that $a - f(a)$ is increasing on $[a_1, a_2]$ and cannot therefore be equal to zero at both ends. This contradiction therefore proves uniqueness, as long as we show that indeed $f' < 1$ on $(0, 1)$. Let $a \in (0, 1)$ and denote $b = \beta(a)$ and $a' = \alpha(b)$. Then by the chain rule $f'(a) = \alpha'(b)\beta'(a)$. Differentiating the equations $F(\alpha(b), b) = 0$ and $G(a, \beta(a)) = 0$, we have $\alpha'(b) = \frac{\phi(b)(a'-b)a'}{\phi(a')+\phi(b)}$ and $\beta'(b) = \frac{\phi(a)(a-b)(-b)}{\phi(a)+\phi(b)}$. Both these derivatives are of the form $\frac{\phi(x_1)(x_1+x_2)x_1}{\phi(x_1)+\phi(x_2)}$ with $x_1, x_2 \in (0, 1)$. This function is increasing in x_2 , hence it is smaller than $\frac{\phi(x_1)(x_1+1)x_1}{\phi(x_1)+\phi(1)}$, which is increasing in x_1 , hence smaller than one, which is the value corresponding to $x_1 = 1$. Thus, $f' < 1$ on $(0, 1)$ and the uniqueness is proved.

To find the unique solution, note that by symmetry we expect $a_t = -b_t$. If we impose this condition, we have $\Phi(a_t) + \Phi(b_t) = \Phi(-a_t) + \Phi(-b_t) = 1$. Therefore, we need to solve the equation $a_t = 2\rho\phi(a_t)$ for $a_t > 0$, or equivalently $g(a_t) = 2\rho$, where $g(x) = \frac{x}{\phi(x)}$. As the derivative of ϕ is $\phi'(x) = -x\phi(x)$, the derivative of g is $g'(x) = \frac{1+x^2}{\phi(x)} > 0$ for all x . Moreover, $g(0) = 0$ and $\lim_{x \rightarrow \infty} g(x) = \infty$, hence g is increasing and a one-to-one and mapping of $(0, \infty)$. Thus, if we define $\delta = g^{-1}(2\rho)$, which is the same formula as in (15), we have $g(\delta) = 2\rho$. The solution of (A20) is then

$$a_t = -b_t = \delta, \quad \text{or} \quad A_t = \mu_t + \delta\sigma_t, \quad B_t = \mu_t - \delta\sigma_t. \quad (\text{A21})$$

Thus, the posterior mean satisfies

$$\mu_{t+1, \text{B}} = \mu_t + \delta\sigma_t, \quad \mu_{t+1, \text{S}} = \mu_t - \delta\sigma_t, \quad (\text{A22})$$

which proves the first part of equation (14).

Equation (A22) also implies that $(\mu_{t+1, \mathcal{O}_t} - \mu_t)^2 = \delta^2\sigma_t^2$ for $\mathcal{O}_t \in \{\text{B}, \text{S}\}$. As $a_t = -b_t$, equations (A18) and (A19) imply that $\sigma_{t+1, \mathcal{O}_t}^2 - \sigma_v^2 + \delta^2\sigma_t^2 = \sigma_t^2$. Thus, the posterior variance

satisfies

$$\sigma_{t+1,B}^2 = \sigma_{t+1,S}^2 = (1 - \delta^2)\sigma_t^2 + \sigma_v^2, \quad (\text{A23})$$

which proves the second part of equation (14). \square

Proof of Proposition 4. Recall that the function $g : [0, \infty) \rightarrow [0, \infty)$ is increasing and $\delta = g^{-1}(2\rho)$, with $\rho \in (0, 1)$. Hence, $\delta < g^{-1}(2) \approx 0.647$, and in particular $\delta < 1$. Equation (A23) implies that the public variance evolves according to $\sigma_t^2 = (1 - \delta^2)\sigma_{t-1}^2 + \sigma_v^2$ for any $t \geq 0$ (by convention, $\sigma_{-1} = 0$). Iterating this equation, we obtain $\sigma_t^2 = (1 - \delta^2)^t \sigma_0^2 + \frac{1 - (1 - \delta^2)^t}{\delta^2} \sigma_v^2$. Using $\sigma_* = \frac{\sigma_v}{\delta}$, we obtain $\sigma_t^2 = \sigma_*^2 + (1 - \delta^2)^t (\sigma_0^2 - \sigma_*^2)$, which proves (17). As $\delta \in (0, 1)$, it is clear that σ_t^2 converges monotonically to σ_*^2 for any initial value σ_0 . The bid-ask spread satisfies $s_t = 2\sigma_t \delta$, hence it converges to $2\sigma_* \delta = 2\frac{\sigma_v}{\delta} \delta = 2\sigma_v = s_*$. \square

Proof of Corollary 2. Following the proof of Proposition 4, recall that g is increasing on $(0, \infty)$. Its inverse g^{-1} is therefore also increasing, and $\sigma_* = \sigma_v / g^{-1}(2\rho)$ is decreasing in ρ . The dependence on σ_v is straightforward. \square

Proof of Corollary 3. Conditional on the information at t , each order (buy or sell) is equally likely. Therefore, the change in the public mean $\mu_{t+1, \mathcal{O}_t} - \mu_t$ has a binary distribution with probability $1/2$, which has standard deviation equal to σ_v , which is the fundamental volatility. \square

Proof of Corollary 4. Equation (A23) shows that the public variance σ_t^2 evolves according to $\sigma_{t+1}^2 = (1 - \delta^2)\sigma_t^2 + \sigma_v^2$. Taking the limit on both sides, we get $\sigma_*^2 = (1 - \delta^2)\sigma_*^2 + \sigma_v^2$. Subtracting the two equations above, we get $\sigma_{t+1}^2 - \sigma_*^2 = (1 - \delta^2)(\sigma_t^2 - \sigma_*^2)$, which proves the speed of convergence formula (22) for the public variance. As $\sigma_t^2 - \sigma_*^2 = (\sigma_t - \sigma_*)(\sigma_t + \sigma_*)$, the formula (22) is true for the public volatility as well. Finally, the bid-ask spread is $s_t = 2\delta\sigma_t$, which proves (22) for the bid-ask spread. \square

Proof of Corollary 5. This follows directly from equations (4) and (5) from Proposition 1, making the change of variables from equation (23). \square

Proof of Proposition 5. The only difference from the setup of Section 2 is that after trading at t (but before trading at $t + 1$) the dealer receives a signal $\Delta s_{t+1} = \Delta v_{t+1} + \Delta \eta_{t+1}$. By

notation, just before trading at t , v_t is distributed as $\mathcal{N}(\cdot, \mu_t, \sigma_t)$. We thus follow the proof of Propositions 3 and 4, and infer that after trading at t the dealer regards v_t to be distributed as $\mathcal{N}(\cdot, \mu'_t, \sigma'_t)$, where $\mu'_t = \mu_t \pm \delta \sigma_t$ and $\sigma'^2_t = (1 - \delta^2) \sigma_t^2$.¹⁸ After observing $\Delta s_{t+1} = \Delta v_{t+1} + \Delta \eta_{t+1}$, the dealer computes $\mathbb{E}(\Delta v_{t+1} | \Delta s_{t+1}) = \frac{\sigma_v^2}{\sigma_v^2 + \sigma_\eta^2} \Delta s_{t+1}$ and $\text{Var}(\Delta v_{t+1} | \Delta s_{t+1}) = \frac{\sigma_v^2 \sigma_\eta^2}{\sigma_v^2 + \sigma_\eta^2} = \sigma_{v\eta}^2$. Hence, after observing the signal, the dealer regards v_{t+1} to be distributed as $\mathcal{N}(\cdot, \mu_{t+1}, \sigma_{t+1})$, with

$$\mu_{t+1} = \mu'_t + \frac{\sigma_v^2}{\sigma_v^2 + \sigma_\eta^2} \Delta s_{t+1}, \quad \sigma_{t+1}^2 = \sigma'^2_t + \sigma_{v\eta}^2 = (1 - \delta^2) \sigma_t^2 + \sigma_{v\eta}^2. \quad (\text{A24})$$

The recursive equation for σ_t is the same as (A23), except that instead of σ_v we now have $\sigma_{v\eta}$. Then, the same proof as in Propositions 3 and 4 can be used to derive all the desired results.

Note that equation (26) implies that the change in public mean is $\Delta \mu_{t+1} = \pm \delta \sigma_t + \frac{\sigma_v^2}{\sigma_v^2 + \sigma_\eta^2} \Delta s_{t+1}$. Thus, in the stationary equilibrium, $\text{Var}(\Delta \mu_{t+1}) = \delta^2 \sigma_*^2 + \frac{\sigma_v^4}{(\sigma_v^2 + \sigma_\eta^2)^2} (\sigma_v^2 + \sigma_\eta^2) = \sigma_{v\eta}^2 + \frac{\sigma_v^4}{\sigma_v^2 + \sigma_\eta^2} = \sigma_v^2 = \text{Var}(\Delta v_{t+1})$. This verifies the result in Appendix B that in any stationary filtration problem the variance of the change in public mean must equal the fundamental variance. Moreover, the half spread is equal to $\delta \sigma_* = \sigma_{v\eta}$, which does not depend on the informed share ρ . \square

Appendix B. Stationary Filtering

We show that in a filtration problem that is stationary (in a sense to be defined below) the variance of value changes is the same as the variance of the public mean changes. Let v_t be a discrete time random walk process with constant volatility σ_v . Suppose each period the market gets (public) information about v_t . Let \mathcal{I}_t be the public information set available at time t . Denote by $\mu_t = \mathbb{E}(v_t | \mathcal{I}_t) = \mathbb{E}_t(v_t)$ the public mean at time t , i.e., the expected asset value given all public information. This filtration problem is called *stationary* if the public

¹⁸The sign \pm is plus if a buy order is submitted at t , and minus if a sell order is submitted at t .

variance is constant over time:

$$\text{Var}_t(v_t) = \text{Var}_{t+1}(v_{t+1}). \quad (\text{B1})$$

The next result gives a necessary and sufficient for the filtration problem to be stationary.

Proposition 6. *The filtration problem is stationary if and only if*

$$\text{Var}(v_{t+1} - v_t) = \text{Var}(\mu_{t+1} - \mu_t).$$

Proof. Since $\mu_t = \mathbb{E}_t(v_t)$, we have the decomposition $v_t = \mu_t + \eta_t$, where η_t is orthogonal on the information set \mathcal{I}_t . Moreover, $\text{Var}(\eta_t) = \text{Var}_t(v_t)$. Similarly, $v_{t+1} = \mu_{t+1} + \eta_{t+1}$, and $\text{Var}(\eta_{t+1}) = \text{Var}_{t+1}(v_{t+1})$. Thus, the stationary condition reads $\text{Var}(v_{t+1} - \mu_{t+1}) = \text{Var}(v_t - \mu_t)$. We can decompose $v_{t+1} - \mu_t$ in two ways:

$$\begin{aligned} v_{t+1} - \mu_t &= (v_{t+1} - \mu_{t+1}) + (\mu_{t+1} - \mu_t) \\ &= (v_{t+1} - v_t) + (v_t - \mu_t). \end{aligned} \quad (\text{B2})$$

We verify that these are orthogonal decompositions. The first condition is that $\text{cov}(v_{t+1} - \mu_{t+1}, \mu_{t+1} - \mu_t) = 0$, i.e., that $\text{cov}(\eta_{t+1}, \mu_{t+1} - \mu_t) = 0$. But η_{t+1} is orthogonal on \mathcal{I}_{t+1} , which contains μ_{t+1} and μ_t . The second condition is that $\text{cov}(v_{t+1} - v_t, v_t - \mu_t) = 0$. But v_t has independent increments, so $v_{t+1} - v_t$ is independent of v_t and anything contained in the information set at time t . (This is true as long as the market does not get at t information about the asset value at a future time.)

The total variance of the two orthogonal decompositions in (B2) must be the same, hence $\text{Var}(v_{t+1} - \mu_{t+1}) + \text{Var}(\mu_{t+1} - \mu_t) = \text{Var}(v_{t+1} - v_t) + \text{Var}(v_t - \mu_t)$. But being stationary is equivalent to $\text{Var}(v_{t+1} - \mu_{t+1}) = \text{Var}(v_t - \mu_t)$, which is then equivalent to $\text{Var}(v_{t+1} - v_t) = \text{Var}(\mu_{t+1} - \mu_t)$.

□

REFERENCES

Bagehot, W. (1971). "The only game in town." *Financial Analysts Journal*, 27(2), 12–14,22.

- Caldentey, R. and E. Stacchetti (2010). “Insider trading with a random deadline.” *Econometrica*, 78(1), 245–283.
- Chau, M. and D. Vayanos (2008). “Strong-form efficiency with monopolistic insiders.” *Review of Financial Studies*, 21(5), 2275–2306.
- Collin-Dufresne, P. and V. Fos (2015). “Do prices reveal the presence of informed trading?” *Journal of Finance*, 70(4), 1555–1582.
- Collin-Dufresne, P. and V. Fos (2016). “Insider trading, stochastic liquidity and equilibrium prices.” *Econometrica*, 84(4), 1441–1475.
- Foucault, T., M. Pagano, and A. Röell (2013). *Market Liquidity: Theory, Evidence, and Policy*. Oxford University Press, New York.
- Glosten, L. R. and P. R. Milgrom (1985). “Bid, ask and transaction prices in a specialist market with heterogeneously informed traders.” *Journal of Financial Economics*, 14, 71–100.
- Glosten, L. R. and T. J. Putnins (2016). “Welfare costs of informed trade.”
- Kyle, A. S. (1985). “Continuous auctions and insider trading.” *Econometrica*, 53(6), 1315–1335.